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Matrix methods and local properties of reversible one-dimensional cellular automata

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Abstract

The local behaviour of reversible one-dimensional cellular automata is analysed. Based on Perron–Frobenius theory, it is proved by means of irreducible components that the connectivity matrices of reversible automata have a single eigenvalue equal to 1. The idempotent behaviour of such matrices is also proved by Faddeev's algorithm. The decomposition of these matrices into triangular factors is used to find the inverse rule for a given reversible automaton.

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1. Introduction

The concept of cellular automata began with the work of von Neumann [16] on self-reproducing systems. The study of cellular automata has acquired great interest because they are systems whose operation is very simple but are able to yield complex behaviours. Relevant examples are the works by Conway [3] and Wolfram in one-dimensional cellular automata [17].

A special type of cellular automaton is the reversible one, so called because of their capacity to return to previously generated states. Reversible one-dimensional cellular automata have been used for modelling and understanding reversible physical and chemical phenomena [14, 17], as well as for implementing data coding systems [4, 15, 17].

Nevertheless, there are several unresolved aspects in these systems. Numerical calculations by means of matrices representing their local behaviour [8, 11, 18] suggest that these matrices are idempotent and have a single positive eigenvalue equal to 1. This type of analysis has been developed in symbolic dynamics by Boyle [1], Lind and Marcus [7], and for additive reversible automata by Kari [6].

The goal of this paper is to prove that the matrices representing the reversible behaviour fulfil the previous features. We shall also show the relation of these results with the conservation of the number of ancestors in reversible one-dimensional cellular automata. Finally, the triangular decomposition of these matrices will be used to obtain the inverse rule defining the reversible behaviour of these systems.

The paper is organized as follows. Section 2 provides the basic concepts of one-dimensional cellular automata and the presentation of their local behaviour by de Bruijn diagrams and connectivity matrices. We shall explain how every one-dimensional cellular automaton is simulated by another of neighbourhood size 2, therefore we just need to study this type of automaton to include the rest. Section 3 applies the theory of non-negative and irreducible matrices to prove that every connectivity matrix has a single positive eigenvalue equal to 1. This property will be important in analysing the number of ancestors of a given sequence. Using the Cayley–Hamilton theorem and Faddeev's algorithm the idempotent behaviour of such matrices is proved. Section 4 discusses the decomposition of the connectivity matrices into triangular factors and how these factors are used for calculating the inverse rule of every reversible automaton. Section 5 presents an example of the previous concepts and the final section contains some concluding remarks of the paper.

2. Basic concepts of one-dimensional cellular automata

A one-dimensional cellular automaton is a system consisting of a finite set K of states of cardinality k, and a one-dimensional array c representing space, each site of the array or cell being with a state of K. Time advances in discrete steps and the dynamics is given by a local mapping used at every time step. The local mapping has two operators, a neighbourhood of $2r + 1$ cells where r is the neighbourhood radius, and a mapping φ or evolution rule. The evolution rule maps all the sequences of $2r + 1$ cells or neighbourhoods into elements of K. For a given initial configuration c, the evolution rule φ is applied to each neighbourhood of c at the same time, where each neighbourhood overlaps in $2r$ states with its contiguous neighbourhoods. In this way, the evolution rule φ yields a new configuration c'.

This process is repeated indefinitely, obtaining at each time step a new configuration or global state of the automaton. Thus, the evolution rule φ defines a global mapping Φ between configurations. A special type of automaton is that where the evolution rule φ has another inverse rule φ^{-1} such that the global mapping Φ is invertible. This type of automaton is called reversible.

In this paper, finite sequences of states are widely used; for this reason some useful definitions are provided. For a set K of states, the set of sequences of n cells formed by states of K is presented by K^n, and K^* is the set of all the finite sequences of states. For $w_1 \in K^n$, $n \geqslant 2r + 1$, let $\varphi(w_1) = w_2$ be the sequence in K^{n-2r} yielded by the action of φ over each neighbourhood of w_1. If $\varphi(w_1) = w_2$ then w_1 is an ancestor of w_2.

The properties of a reversible evolution rule φ are extensively studied by Hedlund [5]. Suppose that φ and φ^{-1} have the same neighbourhood radius, then the reversible automaton has the following properties:

- Every finite sequence of states has the same number of ancestors as the others have. This property may be called the uniform multiplicity of ancestors.
- The number of ancestors of each finite sequence is k^{2r}.
- The ancestors of every sequence of length equal or greater than $2r + 1$ cells have L initial parts, a single common central part and R final parts.
- The values L and R are known as Welch indices and satisfy $LR = k^{2r}$, that is they fulfil the uniform multiplicity of ancestors.

Reviewing the evolution rule with these properties, we can see whether a one-dimensional cellular automaton is reversible. Using the graphical and matrix representation of this rule by means of the de Bruijn diagrams is one way to do this.

2.1. De Bruijn diagrams

Several papers such as those by McIntosh [9, 10], Nasu [12] and Sutner [13] have used the de Bruijn diagrams for representing and analysing the evolution rule of a one-dimensional cellular automaton. The de Bruijn diagrams are defined as follows:

(i) The nodes of the diagram are all the sequences of $2r$ cells, i.e. the set K^{2r}.
(ii) Let a, b be states of K and w_1, w_2 sequences of K^{2r-1}. Then aw_1 and w_2b are sequences of K^{2r} representing nodes of the de Bruijn diagram. There is an arc from aw_1 to w_2b if $w_1 = w_2$, and it represents the sequence $aw_1b \in K^{2r+1}$ which is a complete neighbourhood of the automaton.
(iii) Every arc is labelled by the state in which the neighbourhood aw_1b evolves according to the evolution rule φ.

In this way, paths in the de Bruijn diagram are sequences of symbols formed in a one-dimensional cellular automaton by its corresponding evolution rule. For $n \in \mathbb{Z}^+$, if different paths represent the same sequence $w \in K^n$ in the de Bruijn diagram, then these paths are different ancestors of w. A matrix representation of the ancestors of w is provided by the de Bruijn diagram. The indices of the matrix are the nodes of the de Bruijn diagram. If n ancestors begin at the node a and finish at the node b, then the entry ab of the matrix is equal to n; in the opposite case the entry ab is zero.

Thus, we obtain a non-negative matrix called the connectivity matrix representing the ancestors of w. There are initially k connectivity matrices, one for each state in K. For a greater sequence, the product of matrices is used to yield the connectivity matrix of the sequence. The sum of elements of each connectivity matrix gives the number of ancestors of its associated sequence.

2.2. Cellular automaton simulated by another of neighbourhood size 2

In the evolution rule, neighbourhoods of $2r+1$ states map into a single one; in other words the ancestors have $2r$ more states. Thus if a sequence has n states then its ancestors have $n+2r$ states. Take all the sequences of $2r$ states and their ancestors, each one of $4r$ states. In this way a mapping from K^{4r} into K^{2r} is defined. Each sequence in K^{2r} can be associated with a single state of a new set S of cardinality $|S| = k^{2r}$.

Therefore, each sequence of K^{4r} is associated also with a single element of S^2, and for a one-dimensional cellular automaton, the original mapping given by $\varphi : K^{4r} \to K^{2r}$ is simulated by another mapping $\tau : S^2 \to S$, i.e. by another automaton of neighbourhood size 2. Of course S has a greater number of elements than K, nevertheless, we just need to study automata of neighbourhood size 2 to understand all the other cases.

2.3. Simulating reversible automata

In order to simulate a reversible automaton by another of neighbourhood size 2, the following procedure is defined:

(i) Select between the evolution rule φ and its inverse rule φ^{-1} the greatest neighbourhood size.

(ii) Represent both φ and φ^{-1} with this neighbourhood size. If some rule has a smaller neighbourhood size, then add redundant states to each neighbourhood to obtain the required length.

(iii) For the evolution rule φ, apply the procedure in section 2.2.

With this procedure, the original rule and its inverse are presented by another pair of invertible rules of neighbourhood size 2, and both the de Bruijn diagram and the connectivity matrices have a simple form.

The nodes of the de Bruijn diagram are the elements of the set K. If they are not overlapping nodes, then every node is linked with all the others. The indices of the connectivity matrices are the states in K. Since $2r + 1 = 2$, hence $2r = 1$ and every finite sequence of states has k ancestors, fulfilling the uniform multiplicity property.

As both the original and the inverse rule have neighbourhood size 2, each state is formed by L initial nodes and R final nodes in the de Bruijn diagram representing φ, with $LR = k$. This is fulfilled by φ^{-1} as well, but in this case every state is formed by R initial nodes and L final nodes in the corresponding de Bruijn diagram, also fulfilling that $RL = k$. A relevant result by Nasu [12] is that both the set of initial nodes forming a given state and the set of final nodes forming another (or perhaps the same) state have one and only one single common node.

2.4. Connectivity matrices in reversible automata

For reversible one-dimensional cellular automata of neighbourhood size 2, the connectivity matrix A of a finite sequence $w \in K^n$, $n \in \mathbb{Z}^+$ has the following properties:

(i) The sum of elements in A is equal to k.

(ii) A is a 0–1 matrix because an element greater than 1 implies more than one path from one node to another forming the same sequence. If it happens, since there is a path from the final node to the initial one in another connectivity matrix B, then there is a sequence $w' \in K^m$, $m > n$ with connectivity matrix AB such that a power of AB has more than k elements, contradicting the uniform multiplicity of ancestors.

(iii) Given the property by Nasu [12], there is a single 1 in the main diagonal of A.

(iv) For $i \neq j$, if the entry $a_{ij} = 1$ in A, then the entry $a_{ji} = 0$. This avoids the existence of distinct ancestors for the sequence composed by repetitions of w. Properties (iii) and (iv) have a stronger consequence. There is a single node which is both an initial and a final node. The other initial nodes are distinct from the rest of final nodes, yielding property (v).

(v) For $i \neq j$ and $a_{ij} = 1$ in A, if $a_{ii} = 0$ then the ith column of A is zero. If $a_{jj} = 0$ then the jth row of A is zero.

Therefore, in reversible automata, a connectivity matrix has L equal nonzero rows and R equal nonzero columns. Experimental observations [8, 11, 18] have indicated that these connectivity matrices have a single positive eigenvalue equal to 1. Now the theory of non-negative matrices will be used to prove this property.

3. Connectivity matrices and uniform multiplicity of ancestors

For a square matrix A of order r, let $p(A) = \lambda^r - a_1\lambda^{r-1} - a_2\lambda^{r-2} - \cdots - a_r$ be its characteristic polynomial. Let λ be an eigenvalue of A and v its corresponding eigenvector, fulfilling that $A\lambda = \lambda v$.

Suppose that the indices of A are the nodes of some graph, and each entry of A is the number of arcs between two nodes. Then, the maximum eigenvalue λ_A of A is useful for knowing how the number of paths of finite length grows. The matrix A is irreducible if for any entry a_{ij} there is an integer $n > 0$ such that $a_{ij} > 0$ in A^n. If $Av = \lambda_A v$ then $A^n v = \lambda_A^n v$ and for $a_{ij} \in A^n$

$$\sum_{j=1}^{r} a_{ij} v_j = \lambda_A^n v_i. \tag{1}$$

If the matrix is irreducible, then the eigenvector v of λ_A is positive [2, 7]. For $c = \min\{v\}$ and $d = \max\{v\}$ we have

$$c \sum_{j=1}^{r} a_{ij} \leqslant \sum_{j=1}^{r} a_{ij} v_j = \lambda_A^n v_i \leqslant \lambda_A^n d. \tag{2}$$

Thus, the sum of elements in A has an upper bound:

$$\sum_{i=1}^{r} \sum_{j=1}^{r} a_{ij} \leqslant \sum_{i=1}^{r} \frac{d}{c} \lambda_A^n = \frac{rd}{c} \lambda_A^n. \tag{3}$$

In equation (3), $\frac{rd}{c}$ is constant, and the sum of the elements in A^n (or the number of paths of length n) is bounded by λ_A^n. So, the way in which the number of paths increases or decreases when n increases is described by the behaviour of λ_A^n. Suppose that A is the connectivity matrix of $w \in K^*$ for a reversible automaton of neighbourhood size 2. Then A^n shows the number of ancestors of the sequence w^n.

In this way, A^n reflects the properties of the reversible behaviour. The sum of elements in A^n is equal to k and A^n holds the properties of the Welch indices in its rows and columns. But if the connectivity matrices are not irreducible, then the eigenvector v of λ_A may have zero elements and $\frac{rd}{c}$ is undefined. Thus, $\frac{rd}{c}\lambda_A^n$ does not correctly determine the behaviour of the ancestors in A. In order to solve this problem, we shall use the irreducible components of the connectivity matrices.

3.1. Irreducible components and $\lambda = 1$

Given a non-negative matrix A, its indices can be rearranged so that A will be composed of diagonal blocks. The blocks A_i of A are irreducible components, and the growth of A^n is defined by the growth of the blocks A_i^n. In order to obtain the irreducible components A_i, we shall obtain the communicating classes of A [7]. For $m, n > 0$, if $a_{ij} > 0$ in A^m and $a_{ji} > 0$ in A^n, then the indices i, j belong to the same communicating class.

Once these classes are detected, the matrix A is reordered by permuting rows and columns such that these classes are diagonal blocks. Since the elements of each class connect to each other, every diagonal block is irreducible. The characteristic polynomial of A does not change by permutations of rows or columns, so the diagonal blocks are used for computing its characteristic polynomial in the following way:

$$p(A) = p(A_1)p(A_2)\cdots p(A_n) \tag{4}$$

where $A_1, A_2, \ldots, A_n$ are irreducible diagonal components. The eigenvalues of A are the eigenvalues of every A_i, and the greatest eigenvalue λ_A defines the growth of A^n.

Lemma 1. *In a reversible one-dimensional cellular automaton of neighbourhood size 2 in* φ *and* φ^{-1}*, every connectivity matrix A has one and only one irreducible component of one element.*

Table 1. Recurrent implementation of Faddeev's algorithm.

$A_1 = A$	$p_1 = \mathrm{Tr}\, A_1$	$B_1 = A_1 - p_1 I$
$A_2 = AB_1$	$p_2 = \mathrm{Tr}\, A_2$	$B_2 = A_2 - p_2 I$
$\vdots$	$\vdots$	$\vdots$
$A_{r-1} = AB_{r-2}$	$p_{r-1} = \mathrm{Tr}\, A_{r-1}$	$B_{r-1} = A_{r-1} - p_{r-1} I$
$A_r = AB_{r-1}$	$p_r = \mathrm{Tr}\, A_r$	$B_r = A_r - p_r I$

Proof. A has one and only one diagonal element $a_{ii} = 1$, therefore i is a communicating class. For $i \neq j$, if $a_{ij} = 1$ then the jth row is zero and a_{ij} does not connect with any other row. For $i \neq m$, if $a_{mi} = 1$ then a_{mi} connects with the ith row but no element in this row connects with an element of the mth row because $a_{ij} = 1$ implies that the jth row is zero. Thus no other elements different from a_{ii} form another communicating class and there is just one irreducible component of one element. □

From lemma 1 the following result is provided:

Corollary 1. *In a reversible one-dimensional cellular automaton of neighbourhood size 2 in both invertible rules, every connectivity matrix has a single positive eigenvalue 1.*

Proof. Since in every connectivity matrix A there is a single irreducible component A_i of one element, its eigenvalue is the value of the element. As $a_{ii} = 1$, hence $\lambda_A = 1$. □

Thus, in a connectivity matrix A there is only one irreducible component A_i, L states reach A_i and R states leave it as the Welch indices indicate. Now we shall explain how $\lambda_A = 1$ defines the growth of A^n.

Theorem 1. *In a reversible one-dimensional cellular automaton of neighbourhood size 2 in both invertible rules, for all $n > 0$ the sum of elements of A^n is defined by $L\lambda_A^n R$ and is equal to k.*

Proof. Since A has only one irreducible component A_i of one element, hence A^n has the same irreducible component A_i^n, where $|A_i^n| = \lambda_A^n$. If there are no more components, then only A_i^n is useful to yield sequences of length n. As L different states reach A_i and R distinct states leave A_i, the sum of elements in A^n is given by

$$L\lambda_A^n R = k\lambda_A^n \qquad \text{since } \lambda_A = 1 \text{ hence } k\lambda_A^n = k$$

for all $n > 0$. □

Thus, the single positive eigenvalue of a connectivity matrix determines the uniform multiplicity of ancestors in a reversible one-dimensional cellular automaton. Another way to prove this is using Faddeev's algorithm, which also shows the idempotent behaviour of the connectivity matrices.

3.2. Faddeev's algorithm and idempotent behaviour

For a square matrix A of order r, Faddeev's algorithm [2] defines a recurrent form to calculate the coefficients of its characteristic polynomial using the trace of A (table 1).

Faddeev's algorithm detects the cycles of length 1, 2 up to r. These cycles are the irreducible components of A, and with them the coefficients of the characteristic polynomial are calculated step by step. For a connectivity matrix A of a reversible automaton of neighbourhood size 2 and k states, this procedure has the following form:

(i) Since there is only a positive diagonal element $a_{ii} = 1$ in A, hence $p_1 = 1$.
(ii) In B_1, the diagonal element b_{ii} is 0 and the other diagonal elements are -1.
(iii) Take the product AB_1 and take the ith row in the matrix A with $a_{ii} = 1$. If j is a nonzero column in B_1 and $i = j$ then $b_{jj} = 0$ and $a_{ii}b_{jj} = 0$. For $0 \leqslant m \leqslant k-1$ and $m \neq j$, if $a_{im} = 1$ then $a_{mj} = 0$ so $b_{mj} = 0$, therefore $a_{im}b_{mj} = 0$.
(iv) For $0 \leqslant m \leqslant k-1$ and $m \neq i$, if $i \neq j$ and $a_{ij} = 0$ then $a_{im}b_{mj} = 0$ because if $a_{im} = 1$ then $m \neq j$ and $a_{mj} = 0$, therefore $b_{mj} = 0$.
(v) If $i \neq j$ and $a_{ij} = 1$ then $a_{ii}b_{ij} = 1$ because $b_{ij} = a_{ij}$; but $a_{ij}b_{jj} = -1$ and both terms nullify each other. For $0 \leqslant m \leqslant k-1$, $m \neq i$ and $m \neq j$, if $a_{im} = 1$ then $a_{mj} = 0$ and $b_{mj} = 0$, thus $a_{im}b_{mj} = 0$.
(vi) The same happens for the $L-1$ remaining nonzero rows in A, because they are copies of the ith row with $a_{ii} = 1$. Thus $AB_1 = A_2 = 0$, therefore $p_i = 0$ $(i = 2, \ldots, k)$.

Thus another proof for corollary 1 is provided.

Alternative proof of corollary 1. For the characteristic polynomial of A, Faddeev's algorithm shows that $p_1 = 1$ and $p_i = 0$ for $i = 2, \ldots, k$, then the polynomial has the form

$$p(A) = \lambda^k - \lambda^{k-1} = \lambda^{k-1}(\lambda - 1) = 0. \tag{5}$$

Equation (5) shows an eigenvalue $\lambda_A = 1$ and another eigenvalue $\lambda = 0$ of multiplicity $k-1$. □

Faddeev's algorithm also shows the idempotent behaviour of the connectivity matrices.

Theorem 2. *In a reversible one-dimensional cellular automaton of neighbourhood size 2 in both invertible rules, every connectivity matrix A is idempotent.*

Proof. Let us consider the characteristic polynomial $p(A)$ given in equation (5). The Cayley–Hamilton theorem states that every square matrix obeys its characteristic polynomial [2]. In this way $A^k - A^{k-1} = 0$, i.e. $A^k = A^{k-1}$ and therefore A is idempotent. □

4. Obtaining the inverse rule by decomposition in triangular factors

Another relevant question is whether the connectivity matrices are useful to find the inverse rule. For this reason, the decomposition of a matrix A in triangular factors will be used.

4.1. Decomposition in triangular factors

Let r be the rank of A, Gauss's elimination applied to A yields an upper triangular matrix T_s with r nonzero rows [2]. In order to obtain T_s, some operations are performed between rows of A. Each operation is defined by TA, where T is a lower triangular matrix. Thus Gauss's elimination is taken as the successive product of matrices, yielding another lower triangular matrix T_i which represents all the operations over A:

$$T_s = T_i A. \tag{6}$$

T_i is nonsingular [2] and has an inverse T_i^{-1}, hence the following result is provided:

$$T_i^{-1} T_s = T_i^{-1} T_i A = A \tag{7}$$

where T_i^{-1} is also a lower triangular matrix.

4.2. Factorizing the connectivity matrices

Since these matrices have L equal nonzero rows and each matrix has a single irreducible component of one element, their rank is 1. Given a connectivity matrix A, its indices are reordered such that the diagonal element 1 is in the first row. As the reordered matrix A keeps its L equal nonzero rows, its factor T_s has a single nonzero row.

In order to take the reordered matrix A into T_s, the first row of A must eliminate the other $L-1$ nonzero rows. Since they are equal, the constant to eliminate the rows is -1. The first row is subtracted from the other nonzero rows, yielding T_s. Thus T_i has diagonal elements 1 and the constant -1 is placed in the first column of T_i in the same positions as the nonzero rows of A:

$$T_i A = T_s.$$

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ -1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ -1 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \end{pmatrix}$$

Matrix T_i	Matrix A	=	Matrix T_s
$L-1$ rows with -1 in the first column	L identical nonzero rows		1 nonzero row

Given T_i, its inverse T_i^{-1} must eliminate the elements -1 from the first column of T_i. Thus T_i^{-1} copies each row of T_i, therefore T_i^{-1} has a main diagonal of 1s. The first row of T_i must be subtracted from the other nonzero rows in T_i. In this way, T_i^{-1} has elements equal to 1 in its first column in the same positions where the elements equal to -1 are in the first column of T_i:

$$T_i^{-1} T_s = A.$$

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ 1 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & \vdots \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \\ 1 & \cdots & 1 & \cdots \\ \vdots & & \vdots & \end{pmatrix}$$

Matrix T_i^{-1}	Matrix T_s	=	Matrix A
L rows with 1 in the first column	1 nonzero row		L identical nonzero rows

4.3. Finding the inverse rule

A connectivity matrix A is factorized into two triangular matrices T_i^{-1} and T_s. For a given sequence, the first row in T_s shows the different final states of the ancestors. The same happens to the first column of T_i^{-1}, it shows the initial states of the ancestors of the sequence.

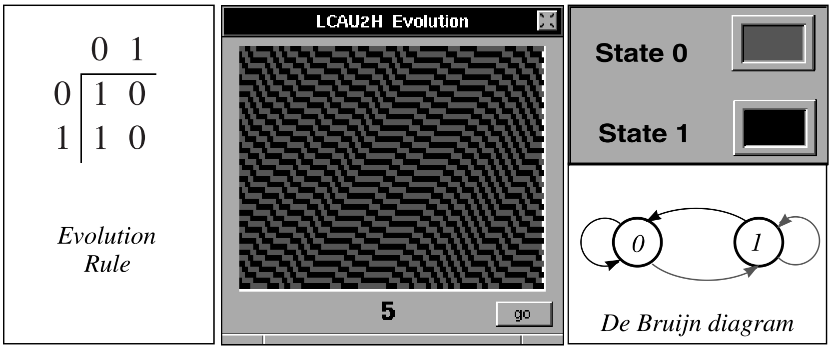

Figure 1. Reversible automaton of neighbourhood size 2.

Then, the first column from T_i^{-1} and the first row from T_s have all the necessary information to find the inverse rule:

(i) Obtain the connectivity matrix of each state.
(ii) Reorder each matrix such that the diagonal element 1 is in the first row.
(iii) Factorize every connectivity matrix into its triangular factors.
(iv) For each state, take the first row v_s from T_s and the first column v_i from T_i^{-1}.
(v) For each state, rearrange v_s and v_i in the original order, that is rearrange its indices from 0 to $k-1$.
(vi) For each state, in the row v_s replace every 1 by the value of its index.
(vii) For every neighbourhood of two cells, multiply the row v_s of the left neighbour by the column v_i of the right neighbour. The result is the state to which the neighbourhood evolves in the inverse rule.

This procedure allows us to find the inverse rule of a reversible one-dimensional cellular automaton simulated by another of neighbourhood size 2. Since the endings of the ancestors intersect in one element [12], the product $v_s v_i$ provides the inverse evolution of each neighbourhood

5. Illustrative example

Take the following reversible automaton of two states and neighbourhood size 2 in figure 1, the connectivity matrix of state 0 shows an eigenvalue $\lambda_A = 1$ and another $\lambda = 0$ of multiplicity 1:

$$\begin{vmatrix} -\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda = \lambda(\lambda - 1) = 0.$$

The idempotent behaviour of the connectivity matrix is shown by its characteristic polynomial (equation (8)):

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}^{k-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{8}$$

For each state, the triangular factors from the connectivity matrices are in table 2. Take the first rows from the matrices T_s, rearrange them in their original order and replace the 1 elements by their corresponding indices. Then multiply every row by every first column of each matrix T_i^{-1}. Thus the inverse mapping of each neighbourhood is obtained (table 3), in this way, the inverse rule of the reversible automaton is presented in figure 2.

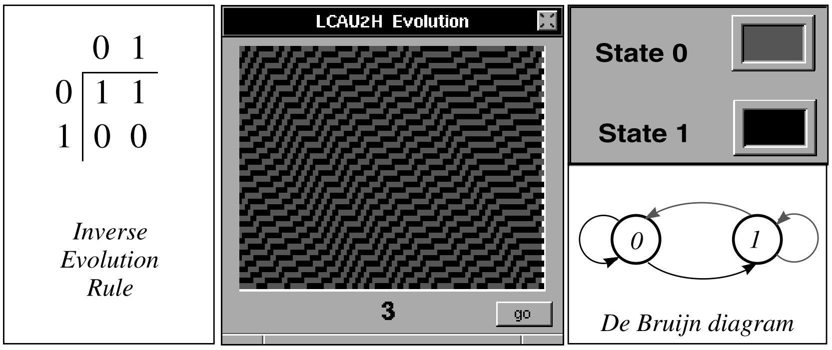

Figure 2. Inverse rule of the automaton in figure 1.

Table 2. Connectivity matrices from the states 0 and 1.

State	Connectivity matrix	Rearrange matrix	T_i^{-1}	T_s
0	$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
1	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Table 3. Inverse mapping of each neighbourhood.

Neighbourhood	Product	Neighbourhood	Product
00	$(0 \quad 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$	01	$(0 \quad 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1$
10	$(0 \quad 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$	11	$(0 \quad 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$

6. Concluding remarks

The Classical theory of matrices, in particular some results such as the Cayley–Hamilton theorem and well-known matrix methods such as Faddeev's algorithm and the decomposition in triangular factors are useful to find important properties of the local behaviour in reversible one-dimensional cellular automata. Besides, the representation of every automaton by another of neighbourhood size 2 is a very convenient form to analyse reversible automata. A further work is to apply other theoretical results about eigenvalues and Jordan normal forms to obtain more features of the local behaviour. This paper has only used the matrix representation of de Bruijn diagrams; a straightforward application of graph theory results may be useful to obtain more properties of these diagrams.

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